Mathematical counterfactuals with number-theoretic antecedents and extra-mathematical explanation

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Abstract

A proposal by Baron, Colyvan, and Ripley to extend the counterfactual theory of explanation to include counterfactual reasoning about mathematical explanations of physical facts is discussed. Their suggestion is that the explanatory role of mathematics can best be captured counterfactually. This paper focuses on their example with a number-theoretic antecedent. Incorporating discussions on the structure and *de re* knowledge of numbers, it is argued that the approach leads to a change in the structure of numbers. As a result, the counterfactual is not about the *natural* numbers anymore. Linking the antecedent and consequent of the counterfactual also becomes problematic.

Keywords mathematical explanation · counterfactual theory of explanation · empirical phenomena · Peano numerals · number-theory · counterpossibles

1 Introduction

In their (2017) article, Baron, Colyvan, and Ripley provide an account of counterfactual reasoning about mathematical explanations of physical facts, also called 'extra-mathematical explanations'. They aim to extend the counterfactual theory of explanation to include those. Extra-mathematical explanations are explanations in which the mathematics contributes to the scientific explanations themselves, and purportedly open the door for an argument for mathematical platonism. In their (2020) article they expand on their account to include intra-mathematical explanations, that is, mathematical explanations of mathematical facts.

In the philosophy of explanation, counterfactual reasoning is rarely geared towards extra-mathematical explanations, that is, to mathematical explanations of empirical phenomena. Although recently, for instance, Knowles and Saatsi (2019) provided an analysis of the role mathematics plays in relation to the explanatory generality of extra-mathematical explanations. In this context, generality pertains to the ability of mathematics to connect seemingly unrelated empirical phenomena under a single schema, which is often considered to be one of the key virtues of the explanatory role of mathematics. This analysis was also carried out within the framework of a counterfactual theory of explanation.

The basic idea of a counterfactual theory of explanation is that some pattern of counterfactual dependence explains how an explanandum would have been different, given a different explanans. Woodward calls this a 'what-if-things-had-been-different question' (Woodward, 2003, p. 11). The explanation thus shows the effect a change of some relevant aspects in the explanans has on the explanandum. More technically, both the explanans and the explanandum contain variables X and Y that can take different values (e.g. x and y), and which are related by some generalization G (Woodward, 2003, p. 203). What is required, is the possibility of at least one of the (values of the variables in the) statements of the explanans being of a different value. This change in value should then lead to a change in some variable in the explanandum, assuming the change fits some generalization employed in the explanation (Reutlinger, 2016, p. 738). As in well-known examples of counterfactual reasoning, such as 'if Suzy had not thrown the rock, the bottle would not have shattered', some parts of the facts are kept fixed. Subsequently, the antecedent is varied, and the implications of the varied antecedent are then considered. In this example one might, as Lewis (1973) suggested, keep the past up to the moment Suzy throws fixed, then vary the fact that Suzy threw the rock (i.e. that she didn't throw the rock in the counterfactual situation), and then look at the implications of the varying of the antecedent (i.e. the bottle did not break in the counterfactual situation).

Baron, Colyvan, and Ripley then apply this general idea to extra-mathematical explanations in which the counterfactuals have a mathematical antecedent. Specifically, they apply it in such a way that the 'what-if-things-had-been-different question' stipulates a change in a mathematical property that occurs in the antecedent of the extra-mathematical explanation. To bring out the explanatoriness of the mathematics, the idea is to evaluate what the effect of this change in the actual mathematics is on the explanandum. To put it in the terms they use, some (mathematical) facts are held fixed, some others are twiddled, and then one looks at the ramifications of the twiddling (Baron, Colyvan, and Ripley, 2017).

In casu, it is applied to the explanation of the prime life cycles of the North American Periodical Cicada and the explanation of the hexagonal shape of honeycomb cells (Baron, Colyvan, and Ripley, 2017). It needs to be emphasized that they explicitly assume there to be genuine cases of mathematical explanations of empirical phenomena, such as the just mentioned explanations, and that their aim is not to discuss them in themselves, but rather to develop a theoretical understanding of such cases (Baron, Colyvan, and Ripley, 2017, p. 2).

The approach is of particular interest, because tweaking the mathematics in the antecedent inevitably leads to impossibilities, given the necessary nature of mathematics. On the standard semantics for counterfactuals, as initially developed by (Stalnaker, 1968; Lewis, 1973), these so called 'counterpossibles' are trivially true. A counterfactual $A \mathrel{\Box\!\!\rightarrow} B$ is said to be true at a world w if and only if it is the case that the closest A-worlds are B-worlds (Berto and Jago, 2019, p. 20). And since there is no A-world in the case of counterpossibles, it follows that there is no closest A-world. Therefore, the universally quantified statement of the form ‘for all closest A-worlds ...’ is trivially true.

The debate on the status of counterpossibles is very much ongoing. Williamson (2018), for example, defends the view that counterpossibles are trivially, or vacuously, true. But Priest (2016) or Berto et al. (2018) defend the idea that counterpossibles can be non-vacuously true. This paper will take no stance in this debate. It is only pointed out that the outcome of this debate is relevant for the evaluation of the counterfactual at hand. After all, in the context of extra-mathematical explanations, the counterfactual has to be non-trivially true, because everything that is trivially true is not essential in an explanation. Given that the tweaking of the mathematics in the antecedent leads to inconsistencies, then, on the standard semantics, the consequent has no role in the evaluation of the truth value of the counterpossible. In other words, whether the consequent is (or contains) B or $\neg B$, in both cases the counterpossible evaluates to true. If the mathematics, and specifically that part of the mathematics that is changed, is to be brought out as carrying explanatory load, then it has to make a difference to the explanation, as is expected of difference makers. But when trivialized, this role cannot be brought out (Baron, 2016, p. 380).[1]

This paper investigates the counterfactual statements in which the antecedent has a mathematical component, and in which the mathematics itself is changed. In particular, the investigation will focus on number-theoretic antecedents, such as is the case in the example of the explanation of the prime life cycles of Periodical Cicadas. Due to the important role the context plays in determining the truth conditions of counterfactuals, the question that is set out to be answered is how to understand what holding the context fixed, whilst tweaking some of the mathematics, could mean in the example using number theory.

Section 2 introduces the example of the counterpossible and points to some immediate worries concerning contradictions that might arise. Section 3 discusses the relevance of such contradictions and argues, by means of Peano Arithmetic, much used in meta-mathematics, that these contradictions point to an underlying difficulty. Section 4 then argues that, in effect, a shift in structure of the natural numbers occurs, resulting from the twiddles in the number-theoretic antecedent. The argument draws on discussions on the standard interpretation of arithmetic and *de re* knowledge of natural numbers. The importance of the idea of being isomorphic to the standard model of arithmetic and of the idea of succession, exemplified in the successor function of the Peano axioms, are discussed. Subsequently it is noticed that for a counterfactual to be the counterfactual of something or other, it needs to be *about* this something or other. The change of structure casts doubt on this.[2] Additionally, and also resulting from the changes in the antecedent, this paper will also question the possibility of the empirical content of the consequent latching on to the antecedent.

[1] Interestingly, in the literature on the logic of so called ‘why-questions’, which are intimately related to explanations, a similar observation is made. For something to be an answer to a why-question it needs to be non-trivially deducible from the explanans (Hintikka and Halonen, 1999, p. 195).

[2] In the context of contradictions arising from changes to the mathematics, the problem of too great a shift in subject is also observed by Baron, Colyvan, and Ripley (2020, pp. 9–10).

2 A mathematical counterfactual

Before turning to the counterpossible we are concerned with, let us first briefly turn to the empirical phenomenon to which it relates. The Periodical Cicada has a remarkable life cycle of either 13 or 17 years. These life cycles are prime, and number theory could explain that, given certain ecological constraints, prime life cycles minimize intersection with other life cycles. In turn, this could be advantageous because they are less likely to intersect with life cycles of, say, predators (Baker, 2005). Whatever the biological status might be of the above explanation of the empirical phenomenon, what Baron, Colyvan, and Ripley are concerned with is the embedding of these kind of extra-mathematical explanations in the general framework of a counterfactual theory of explanation.

They consider the following counterfactual statement (Baron, Colyvan, and Ripley, 2017, p. 6):

(1) If [. . .] 13 [. . .] were not prime, then these Cicadas would not have had 13-year life-cycles lengths.

According to them, this would counterfactually capture the explanatory role played by the mathematics. Since the explanation of the length of the life cycles turns on facts about the number 13, namely that it is a prime number, and also on the number-theoretic theorem about primes and common multiples, namely that prime cycles intersect less compared to non-prime cycles, therefore considering 13 to be not prime would remove the support of the explanation that a 13-year life cycle is an advantageous life cycle to have because it intersects less with other periodical life cycles (given certain ecological constraints).

It is instructive to notice that the above counterfactual is indeed different from a counterfactual like:

(2) If the length of the life cycle were not prime, then these Cicadas would not have developed a life cycle of that length.

In the above counterfactual, the twiddling is not done on the mathematics itself, but on the length of the actual life cycles. The salient difference is that in (1) we are dealing with an impossible antecedent, because mathematical truths are considered to be necessary truths, and in (2) with a possible antecedent, because the length of the life cycle is a contingent truth.

Baron, Colyvan, and Ripley (2017, p. 6) then observe that in order to consider 13 as not being a prime number, it would mean that it is composite. And hence, that it would not only factorize in 1 and 13, but also in some other numbers. It is imagined that, since 13 is not supposed to be prime, it counterfactually also has 2 and 6 as factors. So, the counterfactual could also be stated as follows:

(3) If, in addition to 13 and 1, 13 had the factors 2 and 6, North American Periodical Cicadas would not have 13-year life cycles (Baron, Colyvan, and Ripley, 2017, p. 6).[3]

[3]Note that, strictly speaking, this counterfactual is more precise than counterfactual (1). For the present purposes however, this is of no real importance.

In line with the general idea of a counterfactual theory of explanation, some facts need to be kept fixed and some facts are then varied. In the case of (1) or (3), the idea of Baron, Colyvan, and Ripley is to

> hold all of number theory fixed except for the twiddles to 13 if one is prepared to change the way that multiplication works (Baron, Colyvan, and Ripley, 2017, p. 7).

But if the idea is to only change the factors of the number 13 and keep the rest of number theory as it is, a contradiction would immediately follow. Because, for instance, $2 \cdot 6 = 12$, but since 2 and 6 are now factors of 13 too, also $2 \cdot 6 = 13$ would hold. But at the same time $12 \neq 13$, and so a contradiction arises (Baron, Colyvan, and Ripley, 2017, p. 7).

It looks like either there is no way to continue, or the principle of non-contradiction has to be abandoned and one needs to resort to some paraconsistent logic. However, Baron, Colyvan, and Ripley (2017, pp. 7-8) state that this is not necessarily so. The suggestion is that, when at first 13 is varied (as also having, besides 1 and 13, factors 2 and 6), and a contradiction is obtained, one relaxes some of what was held fixed in the mathematics and varies again. This then goes on until the maximum amount that can be held fixed is found without having a contradiction. This can be done, they suggest, by stipulating a new function other than multiplication, called 'multiplication*', which

> will preserve the same theorems as multiplication, and imbue the natural numbers with the same structure, except for whatever disruption is involved in changing the factors of 13 (Baron, Colyvan, and Ripley, 2017, p. 8).

Although not all of number theory is preserved this way, presumably enough is. This echoes a similar point made by (Berto et al., 2018, p. 704):

> whether a particular chain of reasoning succeeds or fails in supporting the truth of a counterfactual depends on the context, and in particular what truths about the case need to be held fixed to legitimate the reasoning.

Accordingly, Baron, Colyvan, and Ripley (2017, p. 9) state that not all contradictions need to be addressed, but only the relevant ones. The metaphor used is that of ironing out a lump in the carpet away from our area of interest. Similarly, they say, in the case of Suzy there are also bound to be some contradictions. They mention the following: "Suzy moved her arm in a throwing motion, yet the rock did not move? She willed her arm to move, but it didn't?" (Baron, Colyvan, and Ripley, 2017, p. 9)

But we have to be careful here, for it is not immediately clear whether in the just mentioned cases there are bound to be contradictions in the same sense as there are bound to be contradictions in the case of the change in factors of the number 13. Take for instance the example of Suzy moving her arm in a throwing motion, but the rock not moving. Surely the laws of physics are kept the same in this case, because this is relevant to the assessment of the counterfactual 'if Suzy had not thrown the rock, the bottle would not have shattered'. Otherwise nothing could be said about the bottle shattering at all. Or in the case of her willing her arm to move, but the arm did not. Is that a contradiction? And is it relevant for the assessment of the counterfactual at hand? In fact, it could argued that neither of the examples are real contradictions. They are not inconsistent with, for example, classical physics, as long as one is willing to incorporate, say, extra forces. Although perhaps uninteresting from the viewpoint of physical theory, logically speaking there is nothing to prevent one from doing so. To put it in the style of Lewis: although in the case of Suzy a miracle might have occurred, it was however no logical contradiction.

This leaves us with the aspect of the relevance of the contradictions arising from tweaking the the properties of numbers, and whether they need to be addressed at all. Baron, Colyvan, and Ripley (2017, p. 9) state this is follows:

> In a way, our example of the contradiction generated by multiplication (once we suppose that 13 is composite) is a little misleading. After all, it is not clear that any of this is relevant to the counterpossible of interest, so there's no need to even consider the contradictions arising there. We presented the details of how we might iron out such a contradiction by way of example. But, in fact, there is no reason to think that such further contradictions are relevant.

The next section will address the question of the relevance of the contradiction in the case of the mathematical counterfactual at hand.

3 The relevance of the contradiction

Let us first begin with some short remarks on the earlier-mentioned contradiction arising from the twiddling of the number 13, namely that $2 \cdot 6 = 12$ and $2 \cdot 6 = 13$, but that $12 \neq 13$. Depending how the function multiplication* is actually defined, but assuming the rest of number theory is kept the same, we could probably continue from this. For instance, given that multiplication* factors 13 in 1, 2, 6, and 13, this would mean that $2 \cdot 6 = 13$, but then it might also mean that $4 \cdot 3 = 13$ because $(2 \cdot 2) \cdot (6 \div 2) = 13$, if division needs to be changed as a result as well. And so on. A worry that one might have is that a local twiddle reverberates through the whole of number theory (Baron, Colyvan, and Ripley, 2017, pp. 7-8). Since it is not deemed necessary to immediately resort to paraconsistent logic, which will be briefly addressed later, a method is proposed to deal with the ironing out of the (relevant) contradictions. If a contradiction arises due to the twiddling of 13, then some more of the mathematics needs to be varied. If another contradiction arises, then some more of the mathematics is varied still, until one reaches a point without a (relevant) contradiction (Baron, Colyvan, and Ripley, 2017, pp. 7-8). This way, the contradictions should be contained. Apart from describing the general method, the paper does not show how, and if, this would work. How to contain the contradictions arising from the twiddling of 13 in number theory? Remember that multiplication* needs to "imbue the natural numbers with the same structure, except for whatever disruption is involved in changing the factors of 13" (Baron, Colyvan, and Ripley, 2017, p. 8). This therefore raises the further question what the disruptions might be when changing the factors of 13.

Since we are concerned with number theory, a good starting point to evaluate the possibility of the twiddles to multiplication, and the effects such a twiddle might have on other parts of number theory, is Peano Arithmetic. Peano arithmetic is after all meant to define the natural numbers non-circularly. Notions such as 'number' or 'successor' are meant to be captured by the axioms. The language of arithmetic is described as follows, $\mathcal{L}_A = (0, S, +, \cdot, =)$, with 0 a constant, S the function symbol for successor, $+$ and $\cdot$ the function symbols for addition and multiplication respectively, and $=$ the symbol for the equality relation. Peano arithmetic consists of the Peano axioms, and the functions for addition and multiplication (Van Dalen, 2004, p. 87):

Peano axioms

1. $\forall x(0 \neq S(x))$

2. $\forall x \forall y((S(x) = S(y)) \rightarrow x = y)$

3. $\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(S(x))) \rightarrow \forall x \varphi(x)$

Addition

4. $\forall x(x + 0 = x)$

5. $\forall x \forall y(x + S(y) = S(x + y))$

Multiplication

6. $\forall x(x \cdot 0 = 0)$

7. $\forall x \forall y(x \cdot S(y) = x + (x \cdot y))$

Given the Peano axioms, with the Peano numerals as the names for the numbers,[4] it is clear why $\overline{2} \cdot \overline{6} = \overline{12}$ and $\overline{2} \cdot \overline{6} = \overline{13}$ give rise to a contradiction in the first place, i.e. that $\overline{12}$ cannot be equal to $\overline{13}$. Because, generalizing the point, it is the case that the product of the two functions of multiplication and multiplication*, given some numbers m and n, will agree in their outcome eventually. And this violates the requirement of uniqueness, as per axiom 2. Let us be more precise and use $\cdot$ for multiplication, and $\cdot^*$ for multiplication*. Take as an example $m = \overline{6}$ and $n = \overline{2}$ (or vice versa). Then, given the recursive definition of multiplication (axiom 7), it can be deduced that the following holds for multiplication*:

$$\overline{13} = \overline{6} \cdot^* S(\overline{1}) = \overline{6} + (\overline{6} \cdot^* \overline{1}) = \overline{6} + (\overline{6} \cdot \overline{1}) = \overline{12}$$

Considered this way, $\overline{13} = \overline{12}$, unless it would be the case that $\overline{6} \cdot^* S(\overline{1}) \neq \overline{6} + (\overline{6} \cdot^* \overline{1})$.

For the function of multiplication* to be able to deal with this initial contradiction, a first approach could be to define it in an 'exceptionalist' way. For instance, by taking multiplication for all m, n other than $\overline{2}$ and $\overline{6}$, and then define $\overline{2}$ and $\overline{6}$ separately, such

[4] To make the distinction between numbers and numerals explicit, from now on an overline will be added for the numerals. For example, $\overline{0}$ is the name of the number 0.

that $\overline{2} \cdot^{*} \overline{6} = \overline{13} = \overline{6} \cdot^{*} \overline{2}$. Accommodating this in the new function of multiplication*, the recursive step in the axiom of multiplication would then have to be replaced by the following:

$$\forall x \forall y(\neg((x = \overline{6} \wedge y = S(\overline{1})) \vee (x = \overline{2} \wedge y = S(\overline{5}))) \rightarrow (x \cdot S(y) = (x + (x \cdot y))))$$

However, this also means that $\overline{6} \cdot S(\overline{1}) = \overline{6} + (\overline{6} \cdot \overline{1})$ is not provable anymore (just as we have seen in the deduction for multiplication* above). And this in turn means that all resulting models of arithmetic (or rather, arithmetic*) are not isomorphic to the standard interpretation of arithmetic, i.e the arithmetic about the natural numbers, anymore ((Boolos et al., 2007, p. 283)).The next section will address the implications.

Alternatively, to keep the otherwise intended interaction between multiplication, addition, and succession the same, it can be observed that multiplication is defined in terms of addition. So, when multiplying, one applies the function of addition as well. If the derivation of the above mentioned contradiction is to be avoided, then changing multiplication implies changing addition, otherwise $\overline{6} + (\overline{6} \cdot^{*} \overline{1})$ would equal $\overline{12}$, and not $\overline{13}$. Thus, in this approach, the newly suggested function multiplication* requires a different addition function, say addition* (or +*), to deal with the stipulated changes to the number 13 in the mathematical antecedent of the counterfactual. Working through the example, we get the desired equality: $\overline{6} \cdot^{*} S(\overline{1}) = \overline{6} +^{*} (\overline{6} \cdot^{*} \overline{1})$, and therefore $\overline{6} \cdot^{*} \overline{2} = \overline{6} +^{*} \overline{6}$, and $\overline{13} = \overline{13}$.

Of course, the changes to addition* also need to be accommodated in the recursive definition for addition itself (axiom 5): $\overline{6} +^{*} S(\overline{5}) = S(\overline{6} +^{*} \overline{5})$. But this leads to $\overline{6} +^{*} \overline{6} = S(\overline{11})$, and as a result $\overline{13} = \overline{12}$. To prevent the contradiction popping up again here, a change in the successor function is then also needed (called 'succession*' or S^{*}). We get: $\overline{6} +^{*} S^{*}(\overline{5}) = S^{*}(\overline{6} +^{*} \overline{5})$, with which we get $\overline{6} +^{*} \overline{6} = S^{*}(\overline{12})$, and in the end we get the desired equality $\overline{13} = \overline{13}$.

The recursive character of the functions of multiplication (in terms of addition and succession) and addition (in terms of succession) thus results in 'downward' changes. And, effectively, both functions supervene on the successor function. Taken together, and again ignoring the base case as we did before, this means that the recursive step in the function of multiplication will have to accommodate these changes:

$$\forall x \forall y(x \cdot^{*} S^{*}(y) = x +^{*}(x \cdot^{*} y))$$

This approach, in contrast to the first approach, requires some changes to be made to some aspects of number theory other than multiplication. Depending on how to decide which contradictions are relevant and which are not, the hope is that one should be able to draw a line somewhere and allay the worry of the local twiddle reverberating through the whole of number theory. Furthermore, and this is a key point, we have to be sure that we speak about the very same thing in the counterfactual situation as in the factual situation. Thus in the case of Suzy, we have to make sure that we are speaking about the selfsame person in both the factual and counterfactual situation. And in the case of the number 13 in multiplication* (with 13 being not prime), we have to make sure we are speaking about the very same number as the number 13 in the function of multiplication. For example, it has to be so that the number 13 in multiplication* still has all the (relevant) other properties of the number 13 in multiplication, for instance of being between 11 and 13. Baron, Colyvan, and Ripley (2017, p. 5), speaking about the number 17 here, state that:

> in exploring what would happen if 17 were composite, there is no possibility that we're talking about something other than 17 itself, because it is simply part of the specification of the case to be considered that we are talking about 17.[5]

This might seem quite trivial, but number theory is somewhat particular in this respect. Indeed, the intention is to specifically speak about the number 13 in the counterfactual situation. But when ironing out the (relevant) contradictions, we have to make sure that we do not do it in such a way that we end up talking about something else than what we intended to talk about in the first place. Peano axiom 3, the induction axiom, is used to show a shift in context. It is clear that it is this particular axiom that makes some property P the case for all elements of the set. In the case of number theory the essential property P of the set is the successor function $S(x)$. In other words, to be a natural number is to be the member of the set with the successor function. Changing an apparent local thing in the multiplication function therefore implies that the natural numbers are not closed under succession anymore. The role of the successor function has special importance in determining the context in which the natural numbers are understood. This will also be addressed in the next section.

4 The ramifications of the local twiddle

With the need to accommodate the changes of the new function multiplication*, either by an 'exceptionalist' definition, or by having to change the the successor function $S(x)$ as a result of the downward changes, we enter murky waters. The worry is that in either case the changes invoked result in a structure different from the structure of the natural numbers, and that this change results in the fact that we end up speaking about something else than the natural numbers. To get a better grasp on what is at stake when we speak about the structure, or context, of the natural numbers, we will first turn to the domain of computation in order to highlight an important observation about the relation between numbers and numerals. Subsequently, we will turn to the topic of *de re* knowledge of numbers. Having set the stage, we will then return to the present discussion on the change in structure resulting from the twiddle in the antecedent of the counterpossible at hand.

4.1 The structure of natural numbers

Computability[6] pertains to functions. Functions themselves relate members of one set uniquely to members of another set. A function is said to be computable if there exists,

[5] Baron, Colyvan, and Ripley (2017, p. 5) compare this to a remark by Kripke (1980, p. 44): "'Possible worlds' are stipulated, not discovered by powerful telescopes. There is no reason why we cannot stipulate that, in talking about what would have happened to Nixon in a certain counterfactual situation, we are talking about what would have happened to him". But it has to be noted here that Nixon could have existed with very different properties, whereas it is not clear at all if this could also be the case for, say, the number 13.

[6] Note that the concept of 'computability' is mostly an intuitive notion, cf Boolos et al. (2007, p. 23)

in principle, a finite mechanical procedure which computes the output values of the function given certain input values (Rescorla, 2015, p. 338). A well-known formalization for computable functions is the Turing machine, which allows for the study of number-theoretic functions. These machines carry out computations by following a set of explicitly defined rules (a procedure). The exact workings of such a machine is not relevant here, but what is important to notice is that these computations are only carried out on the positive integers, which are in turn represented by some notation, i.e. symbols (Boolos et al., 2007, p. 25). As such, numbers themselves are not encountered in the computations. Rather, the computing of number-theoretic functions is done via, or mediated by, the notation employed in the algorithms (Shapiro, 1982, p. 14). The arguments in a computation are given by means of the numerals of some notational system, e.g. unary, binary, decimal, etc. This brings into question what the relation is between numbers and numerals.

At this point, it might be objected that the computation of number-theoretic functions mediated by notation is still computation over number-theoretic functions. After all "[m]ediated computation is still computation" (Rescorla (2007, p. 273). Besides, we are probably mostly interested in number-theoretic calculations, and not so much the notations of those calculations (cf. Rescorla (2007, p. 253) and Shapiro (2017, p. 269)). This, however, rather restates the need for an account of the relation between the syntactic and the numerical. After all, we want to know which number a certain notational representation is about. It is therefore that Kripke (2011, p. 344) states that

> the notion of computability is best seen as one of having a procedure for knowing which *number* is the value of the function [my emphasis].

This implies that at some point, after doing some calculation, the sense of a term should be immediately revelatory. Whatever is begotten at that point is something which stops the buck, as Kripke (2011, p. 261) calls it. This is the case at the point where no further calculation is required to get to the referent of the term. The question is then which numerical terms are buckstoppers? Kripke gives the following example (p. 261). Given some computable function $f(n)$, say 'the square of n', then 'the square of $\overline{3}$' is not immediately revelatory, for some calculation can still be carried out to get to the referent of that description. Or, to put it differently, it still makes sense to ask 'what number is 'the square of $\overline{3}$'? But $\overline{9}$, on the other hand, is a buckstopper (for people used to the decimal system in any case). No further calculations are needed for anyone familiar with (the decimal) numbers. To emphasize the importance of the use of numerals in computations, one notices that in this example the numerals for the numbers are used (i.e. $\overline{3}$ and $\overline{9}$). Compare the similar remark by Boolos et al. (2007, p. 24):

> if the value of the function we are trying to compute is a number, what our computations in fact end with is a numeral for that number.

Given that the numerals denote the numbers, the context in which the question 'what number is this?' is asked matters. Some notational system might be revelatory to some, but not to others. This brings us to the more general problem of *de re* propositional attitudes, specifically as to what names for numbers are eligible for existential quantification in intensional contexts, and which notational system is (or which notational systems are) acceptable.

With regard to this, Ackerman (1978) aims at determining the conditions under which it can be established which arithmetical terms are eligible for existential quantification in intensional contexts, specifically the integers.[7] This matches well with the present investigation, since we are primarily interested in the structure and status of the natural numbers.

She draws on discussions of Quine (1956) and Kaplan (1968) on existential generalization of *de re* and *de dicto* beliefs, in which it was argued that the relation of a name to a person is such that it represents the individual uniquely. Ackerman (1978) now asks whether the same would be the case for arithmetical terms. Thus, the question is whether all arithmetical terms are eligible for existential quantification in doxastic contexts. Similar to the classic sentences used by Quine and Kaplan,[8] she formulates the following two sentences:

(3) $\exists x$ (Ralph believes x is a perfect number)

(4) Ralph believes that $\exists x$ (x is a perfect number)

She applies the same procedure as Kaplan (1968), and substitutes 'the smallest perfect number' for x in 'x is a perfect number'. The first situation supposes that Ralph has heard of perfect numbers, and that he believes that the smallest perfect number is a perfect number. Ackerman concludes that (3) cannot be inferred from this supposed belief: it is possible that Ralph does not know of any specific number that it is perfect. Contrast the just mentioned belief with a second situation in which Ralph believes that 6 is the smallest perfect number (with 6 substituted for x in 'x is a perfect number'). In this case, she argues, the inference to (3) is justified because it is a *de re* belief about a number (Ackerman, 1978, p. 147). The point is that in the first situation, the argument would have carried over content from the *de dicto* sentence to the *de re* sentence, were the inference accepted. According to the counterexample, however, this should not be the case. The situation is different in the second case.

This leads her to the observation that there is something special about what is expressed by numerals as opposed to other names (for numbers). The suggestion is that a numeral

> *directly* specifies the position of its referent in the progression of numbers (Ackerman, 1978, p. 151).

The difference with a term such as 'the smallest perfect number', which in some sense also specifies the position of its referent in the progression of numbers, is that knowing to which position the numeral refers to is simply a matter of understanding the numeral. In the case of a term such as 'the smallest perfect number' some more

[7]Ackerman specifically targets the integers, because she does not have firm intuitions about other kinds of numbers (Ackerman, 1978, p. 145).

[8]For the examples used, see for instance (Quine, 1956, p. 178) and (Kaplan, 1968, p. 192).

mathematical knowledge is needed than what is needed in order to understand the numeral. For one, it has to be understood what a perfect number is. Concerning the specification of the position of the referent of a numeral in the progression of numbers, we remember the observation made at the end of the previous section regarding the fundamental role the successor function appears to play. What is of importance here, as was observed by Ackerman, is the relation between succession and position, in casu the next position. A progression can typically be thought of as an order with an initial element, such as can be seen in the natural numbers: $0, 1, 2, \ldots$.

Given these observations, the question is thus whether it makes sense to speak about changing something to a number in isolation, such as the twiddles to the number 13, if that would mean a change to the successor function. So far, the suggestion is that this is problematic, and that numbers are indeed to be understood holistically. This is why Ackerman observes the following, when she says that the referent of a numeral can be known directly by understanding the numeral as such:

> [numbers] can be understood only in the context of a system of numbers, and knowing and understanding a system of numerals seems to be a matter of knowing how to generate in order the progression of numerals and knowing how to count transitively[9] ... in accord with the progression (Ackerman, 1978, p. 151).

When discussing *de re* propositional attitudes about natural numbers, Shapiro (2017) stresses, like Ackerman (1978), the importance of the successor function in our understanding of natural numbers, and highlights the importance of the successor function on numbers (Shapiro, 2017, p. 279). Take unary notation, where, for instance, the numerals are the sequences of strokes as used in a simple tallying procedure. The number n would then be denoted by n strokes (e.g. the number seven would be |||||||). This is a structured order: one stroke after the other, say, from left to right. This means that, relative to this notational system, the successor function is computable. This is indeed an important aspect of the natural numbers when used for, for example, determining the cardinality of some set in answer to the question 'how many?'. In general, the cardinality is begotten by reciting the numerals of some notational system (Rescorla, 2007, p. 269).

In fact, Shapiro (2017, p. 279) highlights two features of unary notation, such as putting down strokes, of which he thinks that it makes unary notation privileged over other notations. The first is that each numeral for some number n *is* a collection (of strokes) with cardinality n. The intimate relation between the numeral and its corresponding cardinality can be seen directly. The second feature of unary notation is that it displays the order of the collection (as mentioned above: one stroke after the other, say, from left to right). Thus, unary notation is privileged because it "displays both the cardinality and the corresponding ordinal" (Shapiro, 2017, p. 279).

This does of course not mean that all numerals in unary notation are buckstoppers, at least not for ordinary humans. The unary numeral for a very large number would

[9] With transitive counting is meant counting the cardinality of a collection, in such a way that the numerals for the natural numbers are generated in order and each numeral is associated with a unique object (Shapiro, 2017, p. 272).

most likely prompt the question ‘what number is this?’. But this is not only a problem of unary notation. At some point, no matter what notational system is used, there will be some numerals that are not buckstoppers. The numerals simply get too big (Steiner, 2011, p. 165). However, when computability theory is concerned, one idealizes away from the limitations of ordinary human beings (capacity, time, etc.). From the computational point of view, the computor is an idealized human being (Shapiro, 2017, p. 277). Given the lack of restrictions of this kind for such idealized agents, very large numerals are still buckstoppers.

Returning to the issue of the importance of the successor function, and an initial element on which this function operates, it is clear that the Peano numerals display both elements directly. In a sense this should come as no surprise, because the Peano axioms were meant to define (or describe) the natural numbers. They capture the idea of succession in a fundamental sense, and it uses numerals, i.e. the Peano numerals. In fact, the successor function is hard coded, so to speak, in the numerals. This can be easily seen, because the Peano numerals for the sequence of natural numbers are $\overline{0}, S(\overline{0}), S(S(\overline{0})), \ldots$. For every $n > 0$, the number of S's in the Peano numeral $\overline{n}$ has cardinality n. In light of Shapiro's discussion of unary notation, it is seems that putting strokes on paper in an orderly fashion, as seen above, already gives us something similar to the first two Peano axioms. The first stroke represents the number one, two strokes represent the number two, and so on. The number zero is represented by the empty string (Shapiro, 1982, p. 14).

This makes the Peano axioms particularly relevant in the current context, with specific emphasis on the standard induction scheme, axiom (3), which contains the symbol for the successor function S, and the symbol for 0. There is evidence that this scheme plays a key role in the status of the Peano numerals (Heylen, 2014, pp. 20–21): given the standard induction scheme (with added knowledge operator) it can be deduced that the Peano numerals allow for existential quantification in epistemic contexts. In other words, the Peano numerals are canonical. Or, in Kripke's words, they are ‘buckstoppers’. And importantly, without the standard induction scheme one cannot deduce the canonicity of the Peano numerals. Take for instance the earlier mentioned difference between ‘the square of $\overline{3}$’ and $\overline{9}$. Analogous to the epistemic puzzles mentioned earlier, if someone knows that ‘the square of $\overline{3}$’ equals ‘the square of $\overline{3}$’, this does not entail that they know that ‘the square of $\overline{3}$’ equals $\overline{9}$.[10]

The canonicity of the Peano numerals suggests that they are intimately related to *de re* knowledge about natural numbers. The point is that once the induction principle is accepted, the canonicity of the terms follows. Similarly, if instead of the Peano numerals the induction was formulated by, say, $+\overline{1}$, then the canonicity of the following terms would follow: $\overline{0}, (\overline{0}) + \overline{1}, ((\overline{0}) + \overline{1}) + \overline{1}$ (Heylen, 2014, p. 12). If we recall Ackerman's remark that understanding numbers (and a system of numerals) is very closely tied to knowing how to generate a progression, then it appears to be the case that some form of induction is already implicit in our very understanding of numbers: you know the numbers if you are able to get to the next number, starting from zero. This so called ‘in-

[10]Similarly, the importance of a canonical form is also discussed by Lawrence (2021) in relation to algebraic investigations, i.e. determining the value of a variable in equations. The idea here is that an answer needs to be a complete answer, so that, in the context of giving solutions to algebraic equations, the question as to what value a variable has would not be appropriate given that context.

transitive counting' requires an understanding of a recursive procedure for generating the progression (Benacerraf, 1965, p. 50). Understanding a number is to understand its place in a progression. And if this is so, then a number cannot be separated from the system of numbers which it is part of.

Relating these observations to the counterpossible introduced in section 2, the importance becomes clear. Given the counterpossible, the intention is to speak about the natural numbers themselves, thus *de re*. It is not the intention to speak about their descriptions. And because different terms for numbers do not necessarily all lead to *de re* knowledge of numbers, the primary interest is in the canonical terms for the numbers. If one were intent on separating the individual numbers from the system of numbers, then it is not clear how this can be done without losing a basic tenet of our understanding of the natural numbers. After the local twiddle, the sequence would amount to $\overline{0}, S^*(\overline{0}), S^*(S^*(\overline{0})), \ldots$. In reference to Priest (2016, p. 2660), when we imagine that 13 is not prime (and it is), we intend to imagine something about that very number. Change the number system, and you get different numbers, different canonical properties, etc. And if this is correct, then stating something like $\overline{2} \cdot^* \overline{6} = \overline{13}$ results in the fact that the set of elements does not equal $\mathbb{N}$ anymore.

Coming from a different angle, Halbach and Horsten (2005, p. 181) also endorse the view that numbers are given collectively. They approach it from the perspective of the standard interpretation of arithmetic, and specifically the functions of addition and multiplication. To be more precise, the question they are concerned with is not solely about how and why a numeral for some number n indeed refers to the number n, but rather how a single structure for doing arithmetical operations on the natural numbers can be singled out (Horsten, 2012, p. 276). The standard model of arithmetic is the model of arithmetic intended to be about the natural numbers.[11] The objects in the intended models are (the referents of the names for) the natural numbers (Boolos et al., 2007, p. 104). And, according to structuralists, if it is the case that there are two or more models of this theory, then there is an isomorphism between these models instantiating the same structure.[12] The point is that the objects in the standard model constitute a specific structure, namely the structure for doing arithmetical operations on the natural numbers. But none of the models of the newly suggested function multiplication*, with m, n other than $\overline{2}$ and $\overline{6}$ the same as multiplication, and $\overline{2}$ and $\overline{6}$ defined separately, are models of Peano arithmetic, because $\overline{6} \cdot^* S(\overline{1}) = \overline{6} + (\overline{6} \cdot^* \overline{1})$ is not provable anymore (cf. section 3). This means that, therefore, none of the resulting models is isomorphic to the standard interpretation, i.e. the structure differs. From this it follows that, since the standard models are the intended models about the natural numbers, the new structure is not that of the natural numbers.

The counterpossible about the number 13 not being prime is not the only example of a counterpossible regarding the natural numbers given by Baron, Colyvan, and Ripley (2020). In another, but similar, example of a twiddle to the natural numbers, the same

[11] Some might prefer to refer to it as 'standard models', for instance a structuralist, since there can be various models of arithmetic instantiating the same structure. The philosophical question in this respect is then whether isomorphism implies identity.

[12] First-order Peano Arithmetic, however, has non-standard models, i.e. models not isomorphic to the standard model, and in some of which the Peano axioms are satisfied nonetheless. These models are not intended and need to be ruled out.

problem occurs. This time, it is about numbers counterpossibly being even or uneven. Here too, it is very clear that the structure of the numbers is actually changed. Consider the following counterpossible:

> ... twiddle the natural numbers by making it so that none of 503, 504, or 505 is even. The twiddle ramifies as follows: the product of two natural numbers is even (if and) only if one of the numbers is (Baron, Colyvan, and Ripley, 2020, p. 29).

What is mentioned here is supposed to be about the *natural* numbers, just as it was with the counterpossible about 13 not being prime. But given the earlier observations, if 504 is considered not to be even, this means that the standard arithmetical operation of dividing it by 2 cannot be performed on it. One of the ramifications is that the underlying structure does not satisfy the Peano axioms anymore. In other words, the model that results is not isomorphic to the standard model of arithmetic. And since the standard model of arithmetic was supposed to be, or intended to be, about the natural numbers, what results is a structure different from that of the natural numbers. More so, given the standard induction scheme, axiom (3), and the functions for multiplication and addition, it follows that of three consecutive numbers, at least one needs to be even (or uneven). Thus we see that in this case too, the structure is changed and subsequently differs from the structure of the natural numbers.

These observations suggest that changing an apparently local thing, such as 13 factorizing in 2 and 6 as well as in 1 and 13, or making none of 503, 504, 505 even, has the consequence that one ends up talking about two different structures. And this would then mean that the supposed counterfactual is actually not a counterfactual containing a *natural* number, in our main example the prime number 13 such as used in the original function of multiplication. This is so, because the numeral 13 now *means* something different than the numeral in the function multiplication*, resulting from the changes to the structure.[13]

4.2 Connecting the antecedent with the empirical consequent

The focus so far has been on the antecedent of the counterfactual, but the consequent is also of importance. If the meaning of the antecedent changes, how is it then supposed to latch on to the empirical content of the consequent? In a sense, with multiplication*, we lost the tallying procedure for the measuring of the actual life cycles. How to decide on the cardinality of the set of the life cycles? To remind ourselves, a tallying procedure, even as simple as putting strokes on a piece of paper, already gives us something isomorphic to the Peano numerals. And it does so, because it gives you an initial element and a procedure of succession.

[13]A similar idea can already be found in section 6 of (Quine, 1986), titled ‘Change of logic, change of subject’. Now, someone like Williamson (2007) might argue that the focus on succession as a requirement for the subject matter of mathematical counterfactuals is misplaced, at least in an *epistemic context*, and that one could still meaningfully speak about the natural number 13 in the counterfactual situation, analogously to his discussion on modus ponens. Although an inferentialist might argue that modus ponens is constitutive of the meaning of the conditional. Disagreement between experts about such issues can of course occur, but it leaves open the question whether this can occur in the same sense in the case of number theory, because of the close connection between the structure of a number system, the idea of succession, and numbers.

The worry about the latching on of the antecedent and consequent, due to a difference in context between both sides of the counterfactual, is a similar kind of worry to the one raised by Heylen and Tump (2021, p. 3038) in relation to the Enhanced Indispensability Argument (EIA), from which the case of the mathematical explanation of the prime life cycles of the Periodical Cicadas is taken. The EIA is in essence an inference to the best explanation, and to get an inference to the best explanation off the ground one needs to assume the truth of the explanans as well as the explanandum. Therefore the reference to numbers in the explanandum, due to the notion of prime in it, has to be paraphrased away in first-order logic with identity to prevent circularity (Bangu, 2008; Baker, 2009). It is then observed that the context between the explanans and the explanandum differs after this modification. The explanans treats of an entity, i.e. the number 13 (or 17) of which primeness is predicated, whereas the explanandum treats of a plurality, i.e. the collection of objects of the sentence in first-order logic with identity. Therefore, a bridge principle is needed to logically connect the two contexts. Without such a link between the antecedent and consequent, there is no (mathematical) explanation of the empirical datum. It is then suggested that Hume's Principle[14], given some strict but non-trivial requirement, could link the entities to the pluralities (Heylen and Tump, 2021, pp. 3038–3040).

Pertaining to the counterpossible at hand, two issues need to be addressed. The first issue relates to the difference in context between the antecedent and consequent after the change to the antecedent. The second issue relates to the fact that for a counterpossible to be able to be used in an explanation it should be, as mentioned in the introduction, non-trivially true. The antecedent of the suggested counterpossible is, however, in contradiction with the Peano axioms. As a result, since the antecedent harbours a contradiction, the counterpossible comes out as trivially true. As would any other counterpossible. To avoid the triviality, these inconsistent worlds need to be accommodated. In order to do so, it could be suggested that Lewis's account of closeness be extended to encompass, besides possible worlds, also impossible worlds (Baron, Colyvan, and Ripley, 2020, p. 4). And if the changes to the number 13 lead to contradictions given the standard axioms, and these contradictions cannot be contained as hoped, than it might be accepted that a contradiction tolerant logic is needed (Baron, Colyvan, and Ripley, 2020, p. 5). This would allow for some relevant aspect in the relation between the antecedent and the consequent to be maintained in case of inconsistent worlds, without making the counterpossibles trivially true, because it is not the case that everything is true at such a world, including the negation of everything.

A couple of observations about resorting to a paraconsistent logic with regard to the present context need to be made here. First, it is certainly true that a paraconsistent logic would solve the issue of the contradictions. That is, the fact that a local change to a number, given the Peano axioms (containing the symbol for zero, $\overline{0}$, and the successor function, S), result in a number being identical to a successor of a number, and thereby violating the Peano Axioms, will not result in an explosion. The logical contradictions are taken care of. However, not all of the contradictions that arise are in a sense purely logical contradictions. It remains the case that it is possible to, for example, derive

[14]Hume's principle states that "for any concepts F, G, the number of F is identical to the number of G if and only if F and G are equinumerous" (Shapiro, 2000, p. 110). Or formally, $\forall F \forall G(\#F = \#G \leftrightarrow F \cong G)$ (Boolos, 1998, p. 139).

that $\overline{0} = \overline{1}$.[15] This in itself could be considered to be a problem of sufficient severity in itself, for it removes the basis of the whole system of natural numbers, i.e. the idea of succession, as discussed in the previous sections. Second, even if a paraconsistent logic could constrain the contradictions, this would not be sufficient to put the concerns about the difference in context aside. The point rather is whether we can speak about the same structure as the *natural* numbers after having counterfactually changed the context. Ironing out the logical contradictions appears to be only one side of the coin.

Of course, the point about resorting to a paraconsistent logic can be seen as a 'limiting case'. Jago (2014), for example, does not resort to a paraconsistent logic, and also offers an account in terms of similarity of possible and impossible worlds. A world with an alternative arithmetic in which one small deviation occurs could be said to be closer to the actual world than a world with an alternative arithmetic in which two (or more than one, in any case) deviations occur. However, such an account does indeed require the acceptance of at least a minimal overlap between antecedent and consequent (cf. the point to be made with regard to the first issue below). But the dispute is just as much about the acceptance of such an account. It is only if one were to accept such an account that the question of closeness becomes relevant. But then it needs to be, non-circularly, shown that contradiction harbouring worlds can be sufficiently close to non-contradiction harbouring worlds to be relevant.

Moving on to the first issue pertaining to the difference in context between the antecedent and consequent, and the need for a criterion to connect the two, another difficulty arises. Due to the change in structure of the antecedent, that is, a structure different than that of the natural numbers, the property P of the consequent, pertaining to the actual 13-year life-cycles lengths, does not find a counterpart in the antecedent. The antecedent now contains $\overline{13^*}$, whereas the consequent still contains $\overline{13}$. Given the requirement that the counterpossible should not be trivially true, there needs to be overlap between the antecedent and consequent. Therefore, the consequent should also contain $\overline{13^*}$, or at least $\overline{13^*}$ in a sense that a relevant connection exists given some property P.

But if this is the case, then the number 13 does not occur in the explanandum anymore. However, the explanandum was based on the empirical data, and because an explanation is post hoc, the explanandum is already a given, so 13 has to occur in it (if not paraphrased in first-order logic). Therefore, to say that the physical structure must twist in line with the changes that were made (Baron, Colyvan, and Ripley, 2017, p. 11) raises the problem of the lost number line and the loss of the tallying procedure for the measurement of the actual life cycles. This was a result from the need to change the function of multiplication to multiplication*. In some sense it could be *said* that the 13 year life cycle factorizes in 6 and 2 (Baron, Colyvan, and Ripley, 2017, p. 11), but how is this '13' begotten? In this case, it is definitely not a tallying of the successive years, since that would already have given us the Peano numerals. This, then, is the problem this account faces. If the intention is to keep number theory the same as usual in the consequent, then, contrary to the invocation of Hume's Principle in the analogy from the EIA, it is not at all easy to see what needs to be invoked in case of the

[15]Interestingly, in intuitionistic logic, $0 = 1$ is used as a known contradiction. Negation can, for example, be defined as $\neg A \to 0 = 1$ (Van Dalen, 2001, p. 225).

counterpossible to ensure a link between the antecedent and the consequent.

5 Conclusion

The main question this paper set out to answer is what the effects of changing a mathematical antecedent of a counterfactual are. More specifically whether, given a number-theoretic counterpossible, we can be certain that we are still speaking about the same numbers in both the factual and the counterfactual case. This is important, because a counterfactual of something or other needs to be *about* this something or other. Drawing on insights external to this debate, namely about the structure of numbers, *de re* knowledge of numbers, and the standard interpretation of arithmetic, it appears that this is not as straightforward as it might seem. The ramifications of the intended local change to a number, such as supposing 13 to not be a prime number, result in a change of structure of the numbers.

Given the discussions on the intended models of arithmetic, i.e. the arithmetical models about the natural numbers, it was argued that a local change to a number leads to a change in structure of the natural numbers. However, this new structure is not isomorphic to that of the standard interpretation anymore. And since the standard models are the intended models about the natural numbers, the new structure in not that of the natural numbers. Also, the special status of Peano axiom 3 was highlighted. With it, the canonicity of the Peano numerals can be deduced, and without it this is not possible. This is so, because the induction axiom contains the symbol for zero, 0, and the successor function, $S(x)$. This suggests an intimate connection between Peano numerals and *de re* knowledge of numbers. The counterpossible is of course concerned with numbers *de re*. To put it bluntly, to be a natural number is to be the member of the set with the successor function. After the local change, and the resulting change in structure, this connection is broken, and the antecedent of the counterfactual is not about the *natural* numbers anymore, but about some other numbers with a different structure.

Additionally, resulting from the change in structure of the antecedent of the counterpossible, it was argued that, given that the consequent now differs in structure from it, the latter does not connect to the former.

Some might argue that, in the end, none of the ramifications of the local twiddle are of real importance. For example, because the conversational context would exclude those ramifications, just as certain irrelevant ramifications in ordinary counterfactuals are excluded by the context. However, given the central importance of the idea of succession in our understanding of the natural numbers, i.e. in relation to the holistic aspect of our understanding of numbers, the analogy between the two situations cannot be taken for granted. As such, an account of counterfactuals with number-theoretic antecedents should take into account results from discussions on *de re* knowledge of natural numbers and the standard interpretation of arithmetic. The suggestion is that the way the context stops the corrosive explosion of ramifications in the case of ordinary counterfactuals, such as in the case of Suzy, might not be available in the case of number-theoretic counterfactuals, such as in the Cicada case. To use the metaphor of the carpet, instead of ironing out a lump from our area of interest, it appears there is no carpet left to be ironed out.

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